Section 5.5: Counting Techniques

$$P(E) = \frac{\text{number of ways E can occur}}{\text{total number of possible outcomes}} = \frac{N(E)}{N(S)}$$

Well, sometimes counting the "number of ways E can occur" or the "total number of possible outcomes" can be fairly complicated. In this section, we'll learn several counting techniques, which will help us calculate some of the more complicated probabilities.

The Multiplication Rule of Counting

Let's suppose you're preparing for a wedding, and you need to pick out tuxedos for the groomsmen. Men's Tuxedo Warehouse has a Build-A-Tux feature which allows you to look at certain combinations and build your tuxedo online. Let's suppose you have the components narrowed down to two jackets, two vest and tie combinations, and three shirt colors. How many total combinations might there be?

A good way to help understand this type of situation is something called a **tree diagram**. We begin with the jacket choices, and then each jacket "branches" out into the two vest and tie combinations, and then each of those then "branches" out into the three shirt combinations. It might look something like this:

Example 1

How many 7-character license plates are possible if the first three characters must be letters, the last four must be digits 0-9, and repeated characters are allowed?

[reveal answer]

The total number of license plates would be:

(# choices for 1st character)•(# choices for 2nd)•etc..

= 26•26•26•10•10•10•10 = 175,760,000

Example 2

Many garage doors have remote-access keypads outside the door. Let's suppose a thief approaches a particular garage and notices that four particular numbers are well-used. If we assume the code uses all four numbers exactly once, how many 4-digit codes does the thief have to try?

Source: Sears

[reveal answer]

Not very many!

total # of codes = (# choices for 1st digit)•(# for 2nd)•etc...

= 4•3•2•1 = 24

Notice that the number of choices decreased by one for each digit, since the four numbers were used only once. You'll often see this described either as the number are chosen "without replacement" or that "repeats are not allowed".

Permutations of n Distinct Objects Taken r at a Time

The number of arrangements of r objects chosen from n objects in which

1. the n objects are distinct,
2. repeats are not allowed,
3. order matters,

is given by the formula $_nP_k = \frac{n!}{(n-r)!}$.

Example 3

Suppose an organization elects its officers from a board of trustees. If there are 30 trustees, how many possible ways could the board elect a president, vice-president, secretary, and treasurer?

[reveal answer]

In this example, we have 30 "items" (trustees), from which we're choosing 4. Using the notation from your text, we want to calculate ${}_{30}P_4$, or

$${}_{30}P_4 = \frac{30!}{(30-4)!} = \frac{30!}{26!} = 30 \bullet 29 \bullet 28 \bullet 27 = 657{,}720$$

Example 4

Suppose you're given a list of 100 desserts and asked to rank your top 3. How many possible "top 3" lists are there?

[reveal answer]

$${}_{100}P_3 = \frac{100!}{(100-3)!} = \frac{100!}{97!} = 100 \bullet 99 \bullet 98 = 970{,}200$$

Combinations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of n objects using $r \leq n$ of them, in which

1. the n objects are distinct,
2. repeats are not allowed,
3. order does not matter,

is given by the formula ${}_nC_k = \dfrac{n!}{r!\,(n-r)!}$.

All right, let's try this new one out.

Example 5

Let's consider again the board of trustees with 30 members. In how many ways could the board elect four members for the finance committee?

[reveal answer]

In this example, we have 30 "items" (trustees), from which we're choosing 4. Unlike in Example 3, order doesn't matter for this example, so we're looking at a combination rather than a permutation.

$${}_{30}C_4 = \frac{30!}{(30-4)! \bullet 4!} = \frac{30!}{26!4!} = \frac{30 \bullet 29 \bullet 28 \bullet 27}{4 \bullet 3 \bullet 2 \bullet 1} = 27{,}405$$

You may notice that this number is quite a bit smaller than in Example 3. The reason is that we don't care about order now, so electing trustees A, B, C, and D for the committee is no different than electing trustees B, C, A, and D. That's different than in Example 3, where we were electing them to particular positions.

Example 6

Suppose you're a volleyball tournament organizer. There are 10 teams signed up for the tournament, and it seems like a good idea for each team to play every other team in a "round robin" setting, before advancing to the playoffs. How many games are possible if each team plays every other team once?

Source: stock.xchng

[reveal answer]

This may not initially seem like a combination, but let's take closer look. We have 10 "items" (teams) from which we're choosing 2. We don't care about order, since team A playing team B is the same as team B playing team A. That's exactly a combination!

$${}_{10}C_2 = \frac{10!}{(10-2)! \bullet 2!} = \frac{10!}{8!2!} = \frac{10 \bullet 9}{2 \bullet 1} = 45$$

Wow, that's a lot of games! That's why most tournaments go with a "pool" structure and split the tournament up into two "pools" of 5.

You can see more on the "round robin" structure for tournaments at Wikipedia.