Section 6.2

Binomial Probability Distributions

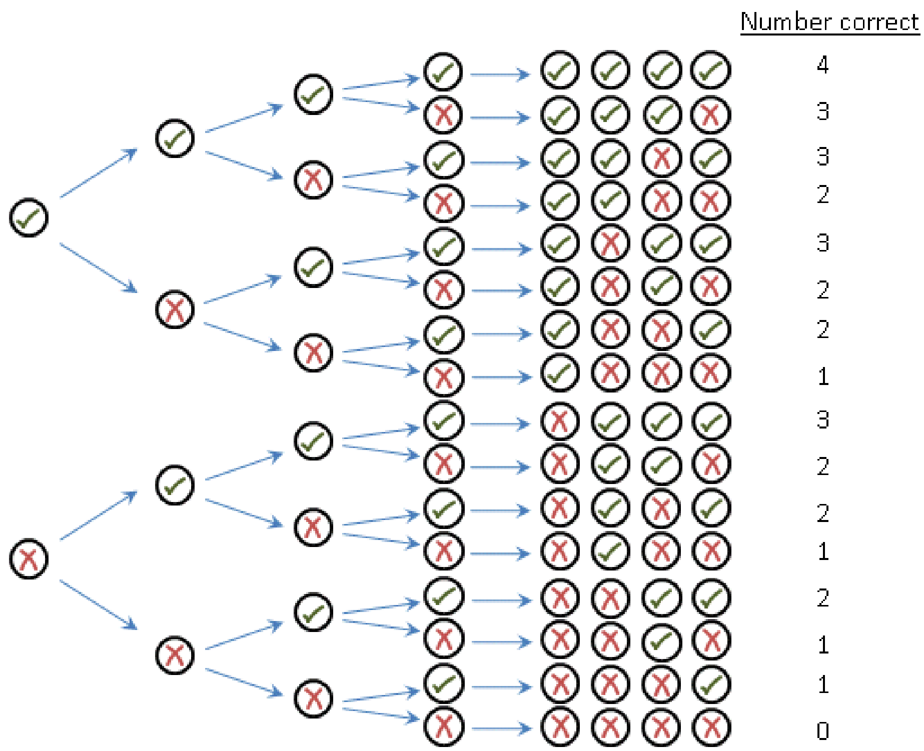
The Binomial Distribution

Once we determine that a random variable is a binomial random variable, the next question we might have would be how to calculate probabilities.

Let's consider the experiment where we take a multiple-choice quiz of four questions with four choices each, and the topic is something we have absolutely no knowledge. Say... theoretical astrophysics. If we let X = the number of correct answer, then X is a binomial random variable because

- there are a fixed number of questions (4)
- the questions are independent, since we're just guessing
- each question has two outcomes - we're right or wrong
- the probability of being correct is constant, since we're guessing: 1/4

So how can we find probabilities? Let's look at a tree diagram of the situation:



Finding the probability distribution of X involves a couple key concepts. First, notice that there are multiple ways to get 1, 2, or 3 questions correct. In fact, we can use combinations to figure out how many ways there are! Since P(X=3) is the same regardless of which 3 we get correct, we can just multiply the probability of one line by 4, since there are 4 ways to get 3 correct.

Not only that, since the questions are *independent*, we can just multiply the probability of getting each one correct or incorrect, so P(✓✓✓✗) = $(3/4)^3(1/4)$. Using that concept to find all the probabilities, we get the following distribution:

x	P(x)
0	$P(0)=\left(\frac{1}{4}\right)^4$
1	$P(1)=4\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3$
2	$P(2)=6\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2$
3	$P(3)=4\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1$
4	$P(4)=\left(\frac{1}{4}\right)^4$

We should notice a couple very important concepts. First, the number of possibilities for each value of X gets multiplied by the probability, and in general there are ${}_4C_X$ ways to get X correct. Second, the exponents on the probabilities represent the number correct or incorrect, so don't stress out about the formula we're about to show. It's essentially:

P(X) = (ways to get X successes)•(prob of success)$^{\text{successes}}$•(prob of failure)$^{\text{failures}}$

The Binomial Probability Distribution Function

The probability of obtaining x successes in n independent trials of a binomial experiment, where the probability of success is p, is given by

$$P(x) = {}_nC_x p^x (1-p)^{n-x}$$

Where x = 0, 1, 2, ... , n

Example 3

Consider the example again with four multiple-choice questions of which you have no knowledge. What is the probability of getting exactly 3 questions correct?

[reveal answer]

For this example, n=4 and p=0.25. We want P(X=3).

We can either use the defining formula or software. The image below shows the calculation using StatCrunch.

n: 4 p: 0.25
Prob(X = 3) = 0.046875
Close Compute

So it looks like P(X=3) ≈ 0.0469

(We usually round to 4 decimal places, if necessary.)

Example 4

A basketball player traditionally makes 85% of her free throws. Suppose she shoots 10 baskets and counts the number she makes. What is the probability that she makes less than 8 baskets?


Source: stock.xchng

[reveal answer]

If X = the number of made baskets, it's reasonable to say the distribution is binomial. (One could make an argument against independence, but we'll assume our player isn't affected by previous makes or misses.)

In this example, n=10 and p=0.85. We want P(X<8).

P(X<8) = P(X≤7) = P(X=0) + P(X=1) + ... + P(X=7)

Rather than computing each one independently, we'll use the binomial calculator in StatCrunch.

n: 10 p: 0.85
Prob(X < 8) = 0.17980352

It looks like the probability of making less than 8 baskets is about 0.1798.

The Mean and Standard Deviation of a Binomial Random Variable

Source: stock.xchng

Let's consider the basketball player again. If she takes 100 free throws, how many would we expect her to make? (Remember that she historically makes 85% of her free throws.)

The answer, of course, is 85. That's 85% of 100.

We could do the same with any binomial random variable. In Example 5, we said that 70% of students are successful in the Statistics course. If we randomly sample 50 students, how many would we expect to have been successful?

Again, it's fairly straightforward - 70% of 50 is 35, so we'd expect 35.

Remember back in Section 6.1, we talked about the mean of a random variable as an expected value. We can do the same here and easily derive a formula for the mean of a binomial random variable, rather than using the definition. Just as we did in the previous two examples, we multiply the probability of success by the number of trials to get the expected number of successes.

Unfortunately, the standard deviation isn't as easy to understand, so we'll just give it here as a formula.

The Mean and Standard Deviation of a Binomial Random Variable

A binomial experiment with n independent trials and probability of success p has a mean and standard deviation given by the formulas

$$\mu_X = np \text{ and } \sigma_X = \sqrt{np(1-p)}$$

Let's try a quick example.

Example 6

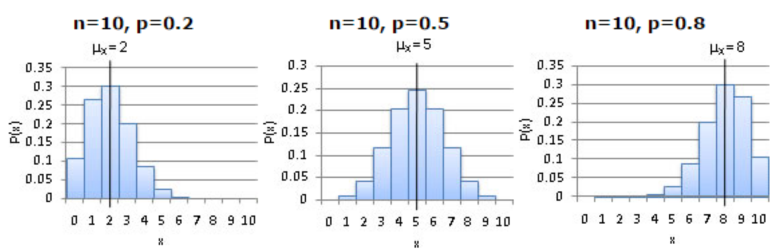
Suppose you're taking another multiple choice test, this time covering particle physics. The test consists of 40 questions, each having 5 options. If you guess at all 40 questions, what are the mean and standard deviation of the number of correct answers?

[reveal answer]

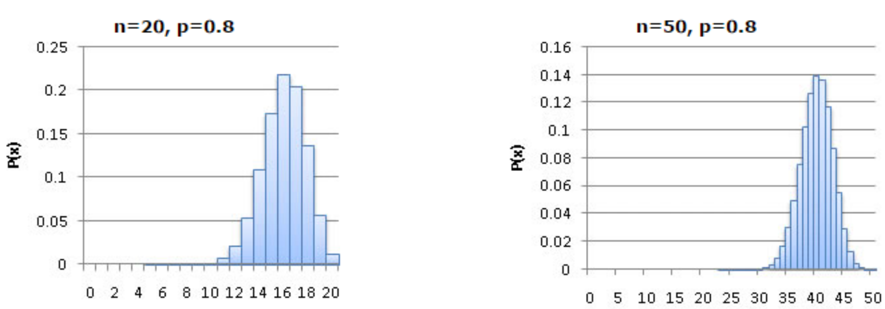
If X = number of correct responses, this distribution follows the binomial distribution, with n = 40 and p = 1/5. Using the formulas, we have a **mean of 8** and a **standard deviation of about 2.53**.

The Shape of a Binomial Probability Distribution

The best way to understand the effect of n and p on the shape of a binomial probability distribution is to look at some histograms, so let's look at some possibilities.



Based on these, it would appear that the distribution is symmetric only if p=0.5, but this isn't actually true. Watch what happens as the number of trials, n, increases:



Interestingly, the distribution shape becomes roughly symmetric when n is large, even if p isn't close to 0.5. This brings us to a key point:

As the number of trials in a binomial experiment increases, the probability distribution becomes bell-shaped. As a rule of thumb, if $np(1-p) \geq 10$, the distribution will be approximately bell-shaped.