Section 9.2: Estimating a Population Mean

Similar to confidence intervals about proportions from Section 9.1, we can also find confidence intervals about population means. Using the same general set-up, we should have something like this:

point estimate ± margin of error

What's our point estimate for the population mean? Why, the *sample mean*, of course! The margin of error will be similar as well:

$$\bar{x} \pm z_{\alpha/2} \cdot \sigma_{\bar{x}}$$

If you recall, we discussed the distribution of $\bar{x}$ in Chapter 8:

The Central Limit Theorem

Regardless of the distribution shape of the population, the sampling distribution of $\bar{x}$ becomes approximately normal as the sample size n increases (conservatively $n \geq 30$).

The Sampling Distribution of $\bar{x}$

If a simple random sample of size n is drawn from a large population with mean μ and standard deviation σ, the sampling distribution of $\bar{x}$ will have mean and standard deviation:

$$\mu_{\bar{x}} = \mu \text{ and } \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

where $\sigma_{\bar{x}}$ is the **standard error of the mean**.

So substituting into the above formula, we get:

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Student's t-Distribution

The so-called Student's distribution has an interesting history. Here's a quick summary taken from Wikipedia:

> The "student's" distribution was actually published in 1908 by W. S. Gosset. Gosset, however, was employed at a brewery that forbade the publication of research by its staff members. To circumvent this restriction, Gosset used the name "Student", and consequently the distribution was named "Student t-distribution".
>
> Source: Wikipedia

Gosset was trying to do research dealing with small samples. He found that even when the standard deviation was not known, the distribution of the sample means was still symmetric and similar to the normal distribution. In fact, as the sample size increases, the distribution approaches the standard normal distribution.

Student's t-Distribution

Suppose a simple random sample size n is taken from a population. If the population follows a normal distribution, then

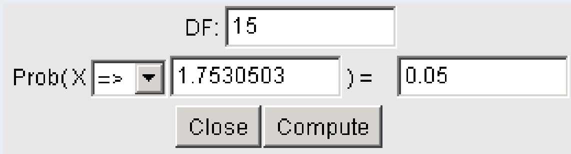
$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

follows a Student's t-distribution with n-1 degrees of freedom.*

Finding Critical Values Using StatCrunch

Click on **Stat** > **Calculators** > **T**

Enter the degrees of freedom, the direction of the inequality, and the probability (leave X blank). Then press Compute. The image below shows the t-value with an area of 0.05 to the right if there are 15 degrees of freedom.



Example

Use the technology of your choice to find $t_{0.01}$ with 14 degrees of freedom

$t_{0.01,\,14} \approx 2.624$

Constructing Confidence Intervals

Before we can start constructing confidence intervals, we need to review some of the theoretical framework we set up in Chapter 8. In particular, the information about the distribution of $\bar{x}$.

The Central Limit Theorem

Regardless of the distribution shape of the population, the sampling distribution of $\bar{x}$ becomes approximately normal as the sample size n increases (conservatively n≥30).

The Sampling Distribution of $\bar{x}$

If a simple random sample of size n is drawn from a large population with mean μ and standard deviation σ, the sampling distribution of $\bar{x}$ will have mean and standard deviation:

$$\mu_{\bar{x}} = \mu \text{ and } \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

where $\sigma_{\bar{x}}$ is the **standard error of the mean**.

Constructing a (1-α)100% Confidence Interval about μ

In general, a **(1-α)100% confidence interval for μ when σ is unknown** is

$$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ is computed with n-1 degrees of freedom.

Note: The sample size must be large (n≥30) with no outliers or the population must be normally distributed.

The **margin of error**, E, in a (1-α)100% confidence interval for μ is

$$E = t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

where n is the sample size.

Confidence Intervals for a Population Standard Deviation

The Chi-Square (X^2) distribution

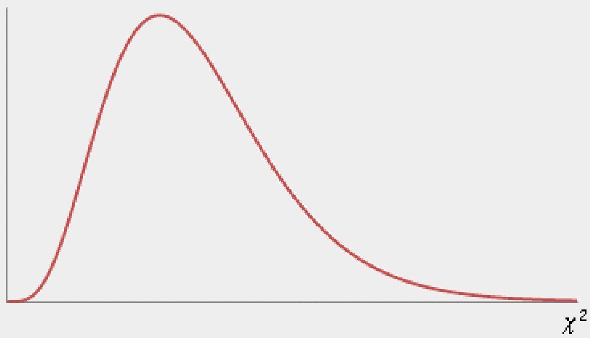
Note: "chi-square" is pronounced "kai" as in sky, not "chai" like the tea.

The Chi-Square (X^2) distribution

If a simple random sample size n is obtained from a normally distributed population with mean μ and standard deviation σ, then

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

has a **chi-square distribution** with n-1 degrees of freedom.



Properties of the X^2 distribution

1. It is *not* symmetric.
2. The shape depends on the degrees of freedom.
3. As the number of degrees of freedom increases, the distribution becomes more symmetric.
4. $X^2 \geq 0$

Finding Critical Values Using StatCrunch

Click on **Stat** > **Calculators** > **Chi-Square**

Enter the degrees of freedom, the direction of the inequality, and the probability (leave X blank). Then press **Compute**.

Example

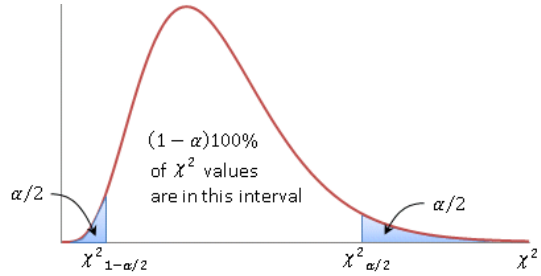
Use the technology of your choice to find $X^2_{0.01}$ with 20 degrees of freedom.

$X^2_{0.01,20} \approx 37.566$

Constructing Confidence Intervals about σ^2 and σ

Now that we have the basics of the distribution of the variable X^2, we can work on constructing a formula for the confidence interval.

From the distribution shape on the previous page, we know that $(1-\alpha)100\%$ of the X^2 values will be between the two critical values shown below.



$$\chi^2_{1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2}$$

If we solve the inequality for σ^2, we get the formula for the confidence interval:

A **(1-α)100% confidence interval for σ^2** is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

Note: The sample *must* be taken from a normally distributed population.

Note #2: If a confidence interval for σ is desired, we can take the square root of each part.

Finding Confidence Intervals Using StatCrunch

1. Click on **Stat** > **Variance** > **One sample**
2. Select **with data** if you have the data, or **with summary** if you only have the summary statistics.
3. If you chose **with data**, click on the variable that you want for the confidence interval. Otherwise, enter the sample statistics.
4. Click on **Next**.
5. Click the Confidence Interval radio button
6. Enter the desired level of confidence and press **Calculate**

The confidence interval should be displayed.

Note: If you need a confidence interval about the population standard deviation, take the square root of the values in the resulting confidence interval.

Here's one for you to try:

Example 4

In Example 3 in Section 9.1, we assumed the standard deviation of the resting heart rates of students was 10 bpm.

heart rate

61	63	64	65	65
67	71	72	73	74
75	77	79	80	81
82	83	83	84	85
86	86	89	95	95

(Click here to view the data in a format more easily copied.)

Use StatCrunch to find a 95% confidence interval for the standard deviation of the resting heart rates for students in this particular class.

[reveal answer]

Using StatCrunch, we get the following result:

95% confidence interval results:

σ^2 : variance of Variable

Variable	Sample Var.	DF	L. Limit	U. Limit
heart_rate	95.333336	24	58.12406	184.49901